Optimal Desaturation of Momentum Exchange Control Systems

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This paper describes a new and systematic approach to the problem of "desaturating" momentum exchange controller devices as used, for instance, in attitude control systems for orbiting spacecrafts. This particular approach is notably efficient because it makes optimal use of the natural external environmental torques acting on the spacecraft (gravity-gradient torques, aerodynamic torques, etc.). Two cases are treated: one where the predominant external disturbing torques are due to well-behaved gravity-gradient torque functions which are assumed completely known a priori; and a more general case where the magnitude and waveform of the disturbance torque functions are unknown (and cannot be measured) but are known to satisfy some given kth-degree linear differential equation. In this latter case, a novel real-time, on-line, physically realizable disturbance estimator is used to obtain an accurate estimate of the external disturbance torques acting on the spacecraft, based on measurements of only the current system state vector. The problem of optimal desaturation of the momentum exchange controller is cast into the mathematical format of a minimal energy optimization problem for a plant represented by a linear, time-varying differential equation with a time-varying forcing term.

I. Introduction

THE use of the angular momentum of spinning inertia wheels to provide control torques for positioning rigid bodies has been of interest for some time.[1–19] Two types of these so-called angular momentum exchange devices have been studied in the literature; the Reaction Wheel system,[2–8] and the Control Moment Gyro (CMG) system.[9–19] Both of these systems allow the resultant momentum vector of the controller to be changed in both magnitude and direction (in the CMG case by forced precession of the individual wheel spin axes, and in the reaction wheel case by forced changes in the individual wheel speeds). The resulting reaction torques exerted on the rigid body are used for control purposes. The control torques produced in this manner can be varied in an extremely smooth fashion and thus, in precision control applications, offer a significant improvement over more conventional on-off type controllers (such as reaction jet devices, etc.).

Fig. 1 depicts a general rigid body with an angular momentum exchange controller. It is assumed that certain types of environmental (external) torques may be acting on the body. Let $\mathbf{H}_v, \mathbf{H}_c$ represent, respectively, the angular momentum of the body and controller measured about the system mass center. The principle of angular momentum states that the absolute time rate of change of the system's total angular momentum vector (body plus controller), measured about the system mass center, is equal to the applied external torque vector $\mathbf{T}_{\text{ex}}$ measured about the system mass center.

$$d/dt(\mathbf{H}) = \mathbf{T}_{\text{ex}}, \quad \mathbf{H} = \mathbf{H}_v + \mathbf{H}_c \tag{1}$$

Received October 13, 1969; revision received June 12, 1970. This research was supported by the NASA Marshall Space Flight Center under Contract No. NAS8-20055, (S.R.C) and No. N.G.L. 01-002-001, (UAH). The authors express their appreciation to C. O. Jones for suggesting the ATM application for the asymptotic estimator discussed in this paper.

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Suppose the rigid body in Fig. 1, whose rotational motion is described by Eq. (1), is a space vehicle (manned or unmanned) which is required to move through space in accordance with some specified mission. In that case, the vehicle mission dictates the vehicle angular momentum required at a given instant. Therefore, the controller must change its momentum vector to maintain the vehicle momentum vector at the desired magnitude and direction. It follows from Eq. (1) that the controller momentum vector must satisfy

$$\mathbf{H}_c(t) = \int_{t_0}^{t} \mathbf{T}_{\text{ex}}(\tau)d\tau + \mathbf{H}_{so} - \mathbf{H}_{vd}(t) \tag{2}$$

where $\mathbf{H}_{so}$ = initial system momentum ($\mathbf{H}_{vo} + \mathbf{H}_{co}$) and $\mathbf{H}_{vd}$ = desired vehicle momentum. Differentiating the balance expression (2) gives

$$d/dt[\mathbf{H}_c(t)] = \mathbf{T}_{\text{ex}} - d/dt[\mathbf{H}_{vd}(t)] \tag{3}$$

Equation (3) expresses the rate at which the controller must be able to change its momentum vector in order to prevent unwanted changes in the vehicle's angular momentum vector. It is assumed hereafter that the controller system is designed with the capability to change $\mathbf{H}_c$ fast enough to accommodate the expected external torques as required by Eq. (3).

While Eq. (3) dictates the rate $d/dt(\mathbf{H}_c)$ at which the controller must be able to change the vector $\mathbf{H}_c$, Eq. (2) indicates the magnitude and direction required of $\mathbf{H}_c$. Even though the initial system momentum $\mathbf{H}_{so}$ and the desired vehicle momentum $\mathbf{H}_{vd}$ may be relatively small, the integral of the external torques may get very large in magnitude. Moreover, in the presence of an external torque function with

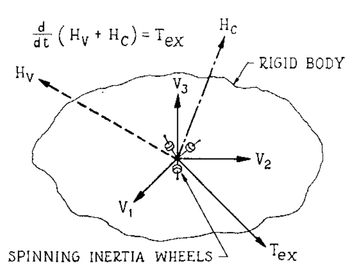

Fig. 1 Momentum exchange control dynamics.

nonzero average value, the time integral of external torques will eventually exceed the momentum magnitude capability, $\|\mathbf{H}_c\|_{\max}$, of any physical controller. When this happens, the angular momentum of the controller is said to be saturated [caused by either the inertia wheels reaching maximum allowable angular speed (reaction wheel system) or the gimbaled spin axes of all of the wheels becoming colinear (CMGs)]. An unsaturated CMG condition is shown in Fig. 2 and a saturated CMG condition is typically illustrated in Fig. 3. A saturated momentum exchange controller can be desaturated, without inducing undesirable angular rotations of the vehicle, only by applying new external torques or by changing the direction and/or magnitude of the existing external torques. In this way the momentum vector $\mathbf{H}_c$ of the controller can be repositioned in direction and/or changed in magnitude (norm) so as to once again have the capacity to satisfy Eq. (2) (i.e., resist the original disturbance torque). This process of restoring the resultant momentum vector of the momentum exchange controller to an unsaturated configuration, Fig. 2, is analogous to the refueling of the controller and is sometimes called momentum dumping.

It is assumed in this study that the desired vehicle angular velocity is zero which implies that the desired vehicle angular momentum $\mathbf{H}_{vd}$ is zero (that is, preferred inertial orientations of the vehicle are specified). However, the techniques presented here may readily be applied to the more general case where the desired vehicle momentum is not zero (for instance, spin stabilized bodies and Earth pointing satellites). In the remainder of this study, the general rigid body in Fig. 1 will be assumed to be an orbiting space station with some specified mission which requires control of the vehicle attitudes.

This paper presents a systematic approach to the problem of efficient desaturation of CMG momentum exchange controllers which makes optimal use of the natural environmental torques acting on the spacecraft (gravity-gradient torques, aerodynamics torques, etc.). Desaturation is defined ideally as a resetting of the inertia wheel gimbal angles of the CMG controller to some preselected values. The present study of the CMG desaturation process is directed at finding the best vehicle maneuvers for achieving desaturation. Thus, the control law for the desaturation process, as developed here, will, in effect, serve as guidance commands to the CMG stabilized system. Two cases will be treated: one where the predominant external disturbing torques are caused by gravity gradients which are assumed completely known a priori; and a more general case, where the magnitude and waveform of the disturbance torque functions are unknown and cannot be measured but are known to satisfy a given kth-degree linear differential equation. The problem is attacked with the tools of modern control theory and is cast into the format of a minimal energy optimization problem.

2. Development of the System Model

In this section it is assumed that the predominant external disturbing torques acting on the vehicle are due to gravity gradients which are completely known a priori. The attitudes $(\theta_1,\theta_2,\theta_3)$ of the vehicle are measured with respect to an inertial coordinate reference frame erected near the desired attitude of the vehicle. As an example, Fig. 4 defines such a frame for a typical sun-pointing mission. If the attitude excursions of the vehicle frame (v) about the reference frame (z) are small, it can be shown[38] that the gravity-gradient torques may be expressed as the sum of two parts

$$\mathbf{T}_g = \mathbf{T}_n(t) + \mathbf{G}(t)\boldsymbol{\theta} \tag{4}$$

where $\mathbf{T}_n$ is a function only of time, $\mathbf{G}(t)$ is a 3×3 time-varying matrix and $\boldsymbol{\theta}$ is the 3-vector $\boldsymbol{\theta}$ = column $(\theta_1,\theta_2,\theta_3)$ representing the small angle deviations of the vehicle frame from the reference frame.

For the special case when the predominant external torques are from gravity gradients, we will set $\mathbf{T}_{\text{ex}} = \mathbf{T}_g$. Substituting Eq. (4) into Eq. (1) yields

$$\dot{\mathbf{H}} = \mathbf{T}_n(t) + \mathbf{G}(t)\boldsymbol{\theta}, \; (\cdot = d/dt) \tag{5}$$

2.1 State Variable Form of the Model

In applications, the desaturation process for a momentum exchange controller is usually required to take place within a prescribed time interval $t_0 \leqq t \leqq T$. Let $\boldsymbol{\sigma}$ denote the desired value of $\mathbf{H}_c(t)$ at the terminal time $t = T$ of the desaturation process. Thus, $\boldsymbol{\sigma} = \mathbf{H}(T)_{\text{desired}}$ since we assume $\mathbf{H}_{vd}(T) = 0$. It is noted that $\mathbf{H}_c(T) = \boldsymbol{\Theta}^{-1}\mathbf{H}_c{}^v(T)$ where $\mathbf{H}_c{}^v$ represents the components of the controller momentum in vehicle coordinates and $\mathbf{v} = \boldsymbol{\Theta}\mathbf{z}$ (see Fig. 4) is the transformation between inertial and vehicle coordinates. A momentum desaturation error vector $\mathbf{H}_e(t)$ may now be defined as

$$\mathbf{H}_e(t) = \mathbf{H}(t) - \boldsymbol{\sigma} \tag{6}$$

and a vehicle attitude error can be expressed as

$$\boldsymbol{\theta}_e(t) = \boldsymbol{\theta}(t) - \boldsymbol{\theta}_d \tag{7}$$

where $\boldsymbol{\theta}_d$ is the desired value of $\boldsymbol{\theta}(T)$. From Eqs. (5–7), the equations of motion may be written,

$$\dot{\mathbf{H}}_e = \mathbf{T}_n(t) + \mathbf{G}(t)[\boldsymbol{\theta}_e + \boldsymbol{\theta}_d] \tag{8}$$

Now, let a set of error state variables for the desaturation process be defined as

$$\begin{aligned} x_1 &= H_{e1} = H_1 - \sigma_1; \quad x_4 = \theta_{e1} = \theta_1 - \theta_{1d} \\ x_2 &= H_{e2} = H_2 - \sigma_2; \quad x_5 = \theta_{e2} = \theta_2 - \theta_{2d} \\ x_3 &= H_{e3} = H_3 - \sigma_3; \quad x_6 = \theta_{e3} = \theta_3 - \theta_{3d} \end{aligned} \tag{9}$$

Using Eqs. (5) and (9) the first derivatives of the state variables in Eq. (9) can be expressed as

$$\begin{aligned} \dot{x}_1 &= [G_{11}(t)x_4 + G_{12}(t)x_5 + G_{13}(t)x_6] + w_1(t) \\ \dot{x}_2 &= [G_{21}(t)x_4 + G_{22}(t)x_5 + G_{23}(t)x_6] + w_2(t) \\ \dot{x}_3 &= [G_{31}(t)x_4 + G_{32}(t)x_5 + G_{33}(t)x_6] + w_3(t) \\ \dot{x}_4 &= u_1; \; \dot{x}_5 = u_2; \; \dot{x}_6 = u_3 \end{aligned} \tag{10}$$

where $\mathbf{w}$ = column $(w_1,w_2,w_3) = \mathbf{T}_n + \mathbf{G}\boldsymbol{\theta}_d$, and

$$\dot{\theta}_i = u_i, \quad i = 1,2,3 \tag{11}$$

Equation (10) may be written more compactly as

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}\mathbf{u} + \boldsymbol{\Sigma}\mathbf{w}(t) \tag{12}$$

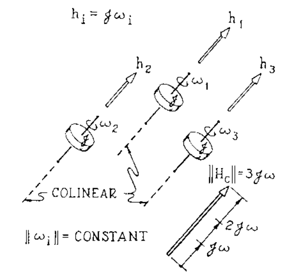

Fig. 3 Saturated CMGs—no reaction torque capability in direction of H_c vector.

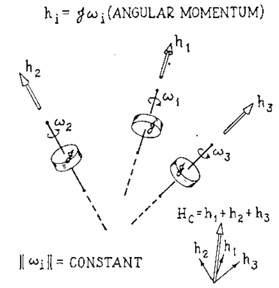

Fig. 2 Unsaturated CMGs—reaction (control) torque capability (i.e., H_c can be changed) in any direction.

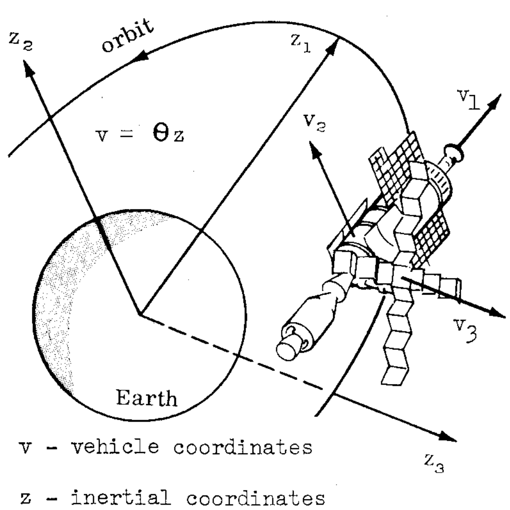

Fig. 4 Coordinate systems for ATM example.

where $\mathbf{x}$ is the 6-vector defined by Eq. (9), $\mathbf{u}$ = column (u_1,u_2,u_3), and $\mathbf{A}(t)$,$\mathbf{B}$, and $\mathbf{\Sigma}$ are matrices defined by

$$\mathbf{A}(t) = \left[\begin{array}{c|c} 0_{3\times 3} & \mathbf{G}(t) \\ \hline 0_{3\times 3} & 0_{3\times 3} \end{array}\right] \qquad 0_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0_{3\times 3} \\ \hline \mathbf{I}_{3\times 3} \end{bmatrix} \qquad \mathbf{\Sigma} = \begin{bmatrix} \mathbf{I}_{3\times 3} \\ \hline 0_{3\times 3} \end{bmatrix} \tag{13}$$

Equation (12) represents the over-all system model, or plant, for the optimization study. It should be noted in Eq. (11) that the control $\mathbf{u}(t)$ has been chosen as the vehicle attitude *rate*. The physical problem is to find a control policy $\mathbf{u}(t)$ to govern the vehicle attitude rates (during the specified interval from t_0 to T) in a way that will meet the several controller desaturation performance specifications as set forth in the next section. It is recalled that the required vehicle attitude rates $u_i(t) = \dot{\theta}_i(t)$ are obtained physically by forced precession of the various spin axes of the gimbaled inertia wheels as depicted in Fig. 2. The particular type of apparatus used to carry out this forced precession is of no concern in the present study but, in practice, it is usually some sort of electrical torquing motor.

The physical control problem just described will now be translated into the format of a precise mathematical optimization problem.

2.2 Precise Statement of a Mathematical Optimization Problem

The performance specifications on the desaturation process are essentially 4-fold: 1) attain, as close as possible, ideal desaturation; 2) prevent excessive attitude errors $\theta_{ei}(t)$ during the desaturation process; 3) at $t = T$ return the vehicle attitude to the preferred value $\boldsymbol{\theta}_d$; and 4) drive the vehicle rates $\dot{\theta}_i$ to approximately zero at $t = T$. It may also be argued that the total energy required to make the special maneuvers for desaturation is of importance since this energy usually comes from solar panels and should be conserved. Minimizing the integral of the vehicle rotational kinetic energy (a function of the square of the rates) would give credence to this energy consideration and also to excessive attitude errors. That is, minimization of this integral would tend to keep some sort of time average of attitude rate small but, if the behavior of $\boldsymbol{\theta}(t)$ is smooth, this will also discourage excessively large peaks in attitude error as well. For these various reasons the minimization of the integral of vehicle kinetic energy along the path $\mathbf{x}(t)$, $t_0 \leqq t \leqq T$, will be chosen as the best single scalar functional to mathematically represent the control energy and peak attitude error considerations.

All four of the aforementioned specifications are of importance in evaluating the total performance of a desaturation policy. However, in order to utilize contemporary optimal control theory to evaluate the relative goodness of any two different control policies one must identify a single scalar quantity (not four) on which to base a judgment. In the present study, this will be accomplished by summing scalar measures of the four performance specifications with weighting factors indicative of their relative importance. In practical applications, such as the NASA Apollo Telescope Mount (ATM) project which we will discuss in a later section of this paper, it is particularly important to return the vehicle to the reference attitude $\boldsymbol{\theta}_d$ at time T, even if the controller is not completely desaturated $[\mathbf{H}_e(T) \neq \mathbf{0}]$. This is required for the ATM project in order to be prepared for certain mission orbital pointing experiments which begin at terminal time T. Therefore, it is convenient to say that specification 3 is $\mathbf{\Lambda}$ times more important than specification 2 and that specification 1 is $\mathbf{\Delta}$ times more important than specification 2. Now, if appropriate positive definite scalar expressions for each of the four performance specifications are chosen, then the weighted sum will constitute a single scalar expression by which one can judge the relative goodness of total system performance. Assuming that $\dot{\boldsymbol{\theta}}(T) = \mathbf{0}$ is achieved, a convenient positive definite expression for the measure of specification 1) is the square of the momentum error norm at time T. Namely,

$$\|\mathbf{H}_e(T)\|^2 = \mathbf{H}_e(T)'\mathbf{H}_e(T), \text{ ('—denotes transpose)} \tag{14}$$

Similarly, as a measure of specification 3 one can choose the square of the attitude error norm at time T

$$\|\boldsymbol{\theta}_e(T)\|^2 = \boldsymbol{\theta}_e(T)'\boldsymbol{\theta}_e(T) \tag{15}$$

The rotational kinetic energy (k.e.) of the vehicle at any time t is given by (for small attitude deviations)

$$\text{k.e.} = \tfrac{1}{2}\dot{\boldsymbol{\theta}}(t)'\mathcal{J}\dot{\boldsymbol{\theta}}(t) \tag{16}$$

where $\mathcal{J}$ is the positive definite, symmetric 3×3 vehicle inertia matrix (about body geometric axes). As explained previously, the time integral of vehicle rotational kinetic energy is an approximate measure of the expended control energy and peak attitude error. An over-all (total) system performance measure J can now be formed as a weighted sum of Eqs. (14) and (15) and the integral of Eq. (16). This gives the mathematical functional to be minimized as

$$J[\dot{\boldsymbol{\theta}}] = \mathbf{H}_e(T)'\mathbf{\Delta}\mathbf{H}_e(T) + \boldsymbol{\theta}_e(T)'\mathbf{\Lambda}\boldsymbol{\theta}_e(T) + \frac{1}{2}\int_{t_0}^{T} \varphi(t)\dot{\boldsymbol{\theta}}(t)'\mathcal{J}\dot{\boldsymbol{\theta}}(t)dt \tag{17}$$

where $(\mathbf{\Lambda},\mathbf{\Delta}) = 3 \times 3$ symmetric, positive definite weighting matrices and $\varphi(t)$ is a positive scalar function of time chosen so as to achieve $\dot{\boldsymbol{\theta}}(T) \approx \mathbf{0}$.‡ Using the definitions from Eqs. (9) and (11), the functional [Eq. (17)] may be written in more compact form as

$$J[\mathbf{u}] = \frac{1}{2}\mathbf{x}(T)'\mathbf{F}\mathbf{x}(T) + \frac{1}{2}\int_{t_0}^{T} \mathbf{u}(t)'\mathbf{R}(t)\mathbf{u}(t)dt \tag{18}$$

where

$$\mathbf{F} = 2\left[\begin{array}{c|c} \mathbf{\Delta} & 0_{3\times 3} \\ \hline 0_{3\times 3} & \mathbf{\Lambda} \end{array}\right] > 0, \ \mathbf{R}(t) = \varphi(t)\mathcal{J} > 0, \text{ for all } t_0 \leqq t \leqq T$$

The mathematical optimization problem, then, is to find a continuous vector control $\mathbf{u}(t)$,$t_0 \leqq t \leqq T$, which minimizes the functional Eq. (18) subject to Eq. (12) and the given

‡ One should choose $\varphi(t)$ so that $\varphi(t) \approx 1$, $t_0 \leqq t \leqq (T - \rho)$, $\rho > 0$, and $\varphi(t)$ = "appropriately large" for $(T - \rho) < t \leqq T$. For instance, one might choose: $\varphi(t) = 1 + \epsilon[(T + \beta - t)/\rho]^{-k}$, (ϵ,β) = small $> 0, k \gg 1$.

initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$. The times t_0 and T are assumed to be fixed a priori.

3. Solution of the Problem in the Case of Known Disturbances $\mathbf{w}(t)$

In the previous section the particular plant of interest was shown to be described by a sixth-order linear time-varying system of the form

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{\Sigma}\mathbf{w}(t) \quad (19)$$

However, the mathematical theory to be discussed in this and subsequent sections is generally applicable to all systems of the form of Eqs. (19) so that we hereafter consider $\mathbf{x} = n$ vector, $\mathbf{A}(t) = n \times n$ time varying matrix, $\mathbf{B}$ is an $n \times m$ constant matrix, $\mathbf{u}$ is an m vector, $\mathbf{w}(t)$ is a time-varying r vector, and $\mathbf{\Sigma}$ is an $n \times r$ constant matrix, in Eq. (19).

The problem of finding a vector control $\mathbf{u}(t)$ that will minimize Eq. (18) subject to Eq. (19) and the boundary conditions

$$\mathbf{x}(t_0) = \mathbf{x}_0,\ \mathbf{x}(T) = \text{unspecified},\ (t_0,T) = \text{specified} \quad (20)$$

has been thoroughly discussed in Refs. 27–29. In Ref. 27, the solution obtained was labeled unrealizable since it requires advance knowledge of the disturbance function $\mathbf{w}(t), t_0 \leqq t \leqq T$. In the particular problem considered in this section, however, $\mathbf{w}(t)$ is just as well known as the plant parameters in the matrix $\mathbf{A}(t)$. Therefore, in this section $\mathbf{w}(t)$ may be realistically assumed to be a completely known function of time $t_0 \leqq t \leqq T$.

The candidates for the optimal control $\mathbf{u}^*(t)$, can be identified by straightforward use of the Pontryagin Principle.[30] For this purpose, the Hamiltonian is written as

$$H(t,\mathbf{x},\mathbf{u},\mathbf{p}) = \mathbf{p}'[\mathbf{A}(t)\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{\Sigma}\mathbf{w}(t)] + \tfrac{1}{2}\mathbf{u}'\mathbf{R}(t)\mathbf{u} \quad (21)$$

where $\mathbf{p}(t)$ is the costate n vector, and the extremal control is determined as

$$\mathbf{u}^*(t) = -\mathbf{R}(t)^{-1}\mathbf{B}'\mathbf{p}(t) \quad (22)$$

It can be shown[27] that, under appropriate assumptions, the extremal control [Eq. (22)] exists and is unique so that Eq. (22) actually represents the sought optimal (minimizing) control. The vector $\mathbf{p}(t)$ obeys the differential equation

$$\dot{\mathbf{p}} = -\mathbf{A}'(t)\mathbf{p} \quad (23)$$

The corresponding transversality condition, which provides a boundary condition on $\mathbf{p}(t)$, takes the form

$$\mathbf{p}(T) = \text{grad}_{\mathbf{x}}(\tfrac{1}{2}\mathbf{x}'\mathbf{F}\mathbf{x})_{t=T} = \mathbf{F}\mathbf{x}(T) \quad (24)$$

Moreover, $\mathbf{p}(t)$ may be expressed in the first integral form

$$\mathbf{p}(t) = \mathbf{K}(t)\mathbf{x}(t) + \mathbf{h}(t) \quad (25)$$

where $\mathbf{K}(t)$ is an $n \times n$ time-varying symmetric matrix that satisfies the matrix Riccati differential equation

$$\dot{\mathbf{K}}(t) = -\mathbf{K}(t)\mathbf{A}(t) - \mathbf{A}(t)'\mathbf{K}(t) + \mathbf{K}(t)\mathbf{B}\mathbf{R}(t)^{-1}\mathbf{B}'\mathbf{K}(t);\ \mathbf{K}(T) = \mathbf{F} \quad (26)$$

and the vector $\mathbf{h}(t)$ satisfies the linear differential equation

$$\dot{\mathbf{h}}(t) = [\mathbf{K}(t)\mathbf{B}\mathbf{R}(t)^{-1}\mathbf{B}' - \mathbf{A}(t)']\mathbf{h}(t) - \mathbf{K}(t)\mathbf{\Sigma}\mathbf{w}(t);\ \mathbf{h}(T) = \mathbf{0} \quad (27)$$

Thus, from Eqs. (22) and (25), the optimal control law can be written in the form

$$\mathbf{u}^*(t,\mathbf{x}) = -\mathbf{R}(t)^{-1}\mathbf{B}'\mathbf{K}(t)\mathbf{x} - \mathbf{R}(t)^{-1}\mathbf{B}'\mathbf{h}(t) \quad (28)$$

Since the matrix Riccati Eq. (26) is independent of $\mathbf{h}(t)$, it may be solved independently of Eq. (27) starting at $\mathbf{K}(T) =$

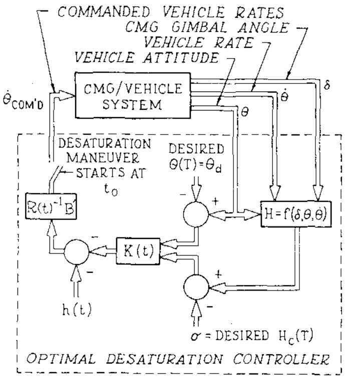

Fig. 5 Block diagram of optimal desaturation control system for case when $w(t)$, $t_0 \leq t \leq T$, is completely known.

$\mathbf{F}$ and integrating backwards to t_0. Then, with $\mathbf{K}(t)$ completely known, Eq. (27) may be solved starting at $\mathbf{h}(T) = \mathbf{0}$ and integrating backwards to t_o. These computations can be done prior to t_o since they do not depend on $\mathbf{x}(t_o)$. Thus, the functions $\mathbf{K}(t),\mathbf{h}(t)$ are functions only of (t_o,T), and the assumed known function $\mathbf{w}(t), t_o \leqq t \leqq T$, and can therefore be completely determined before the problem starts. The block diagram in Fig. 5 illustrates the feedback structure of the optimal desaturation control law of Eq. (28).

If the initial and terminal times (t_o,T) for the desaturation process are to vary during the mission, the gains $\mathbf{K}(t)$ and the forcing function $\mathbf{h}(t)$ must be computed prior to each desaturation interval. If the mission is such that a given portion of the orbit is specified for desaturation (thus defining t_o and T), the quantities $\mathbf{K}(t)$, $\mathbf{h}(t)$ can be entirely precomputed before flight. On examining the time-varying optimal gains for a specific mission, it may prove practical and near optimal to substitute piecewise constant gains for the elements of $\mathbf{K}(t)$ as proposed in Ref. 31.

The general control law established in this Section is directly applicable to the CMG momentum desaturation problem by using the particular definitions in Section 2 for $\mathbf{A}(t)$, $\mathbf{B},\mathbf{\Sigma},\mathbf{w}(t),\mathbf{F}$, and $\mathbf{R}(t)$.

4. Solution of the Problem in the Case of Unknown Disturbances $\mathbf{w}(t)$

The theory discussed in Section 3 shows that the optimal controller requires advance knowledge of the disturbance $\mathbf{w}(t)$.§ If, as discussed in previous sections, the predominant vehicle disturbance torques are from gravity gradients, it is reasonable to treat $\mathbf{w}(t)$ as a known disturbance, since momentum exchange controllers do not expel mass and the vehicle inertia tensor remains essentially constant (and therefore the gravity-gradient torques do not change). If gravity-gradient torques are not the only significant disturbances and if the additional disturbances are not known exactly then it is plausible that, from an engineering point of view, the appropriate thing to do is to make some best estimate¶ $\hat{\mathbf{w}}(t)$ of the disturbance function $\mathbf{w}(t)$ and use that estimate in the control law Eq. (28). In other words, replace $\mathbf{w}(t)$ in Eq. (27) by the estimate $\hat{\mathbf{w}}(t)$, $t_o \leqq t \leqq T$. Several

§ The external torques on the vehicle contribute terms in the matrix $\mathbf{A}(t)$ as well as the forcing term $\Sigma\mathbf{w}(t)$. This section treats the uncertainties in the forcing term $\Sigma\mathbf{w}(t)$ of Eq. (19). It is expected that [with similar uncertainties in $\mathbf{A}(t)$ and $\Sigma\mathbf{w}$] uncertainties in $\Sigma\mathbf{w}(t)$ will be more crucial because there are no feedback paths around these terms; see Eq. (19). The specific effect of uncertainties in $\mathbf{A}(t)$ (commonly called modeling errors) should be studied by simulation.

¶ It can be shown that under certain separability conditions[32] the stochastic version of the optimal control problem treated here breaks into two parts; designing the estimator independently of the control, and designing the controller as if the estimated state is the actual state. In the deterministic version of the problem treated here, we will proceed in the same spirit.

schemes for computing an estimate $\mathbf{w}(t)$ are described in the following paragraphs.

In the most general case of arbitrary unrestricted disturbances, estimates $\hat{\mathbf{w}}(t)$ are most effectively obtained by the use of so-called inverter systems. Suppose the system under study is described by the state Eq. (19) where $\mathbf{x}$ is an n vector, $\mathbf{u}$ is an m vector, $m \leqq n$, and $\mathbf{w}$ is an r vector, $r \leqq m$. It is clear that the matrix Σ may be assumed to have maximal rank, without loss of generality. In this case, an inverter system for $\mathbf{w}(t)$ consists of a physically realizable apparatus for performing approximate differentiation $\tilde{\dot{\mathbf{x}}}(t)$ of the vector $\mathbf{x}(t)$ and solving the equation

$$\hat{\mathbf{w}}(t) = (\mathbf{\Sigma}'\mathbf{\Sigma})^{-1}\mathbf{\Sigma}'[\tilde{\dot{\mathbf{x}}}(t) - \mathbf{A}(t)\mathbf{x}(t) - \mathbf{B}\mathbf{u}(t)];\ \tilde{\dot{\mathbf{x}}} \approx \dot{\mathbf{x}} \quad (29)$$

Equation (29) represents an instantaneous estimator which provides an estimate of only the current value of $\mathbf{w}(t)$. This type of estimator, for arbitrary disturbances, will not satisfy the requirements of the present problem because it does not predict the disturbance over future time as required to solve Eq. (27) for $\mathbf{h}(t)$.

In general, it is not possible to predict the disturbance over a future interval without some prior information about the mathematical properties of the disturbance function $\mathbf{w}(t)$. Therefore, in order to solve the present problem, it is necessary to make some assumptions regarding a priori information about the qualitative and/or quantitative properties of $\mathbf{w}(t)$. One such assumption, which is particularly realistic for the case of disturbance torques acting on orbiting spacecrafts, is described in the next section.

4.1 A Dynamical Model for a Restricted Class of Disturbances $\mathbf{w}(t)$

One way to characterize the general qualitative features of a function $\mathbf{f}(t)$ is to give a differential equation which $\mathbf{f}(t)$ is known to satisfy. This idea is a useful method for characterizing the unknown disturbances $\mathbf{w}(t)$ which act on orbiting spacecraft systems of the form of Eq. (12) since such disturbance functions are usually regular in nature and close approximations to $w_i(t)$ can be obtained by summing certain weighted combinations of the elementary functions: $t^k, e^{\pm\alpha t}$, $\sin\omega t$, $t^k e^{\pm\alpha t} \sin\omega t$, etc. In other words, the actual physical disturbance approximately satisfies a linear differential equation. Thus, in this section, $\mathbf{w}(t)$ is assumed to be accurately modeled by the dynamical disturbance process

$$\mathbf{w}(t) = \mathbf{P}\boldsymbol{\gamma}(t) \quad (30)$$

$$\dot{\boldsymbol{\gamma}}(t) = \mathbf{D}\boldsymbol{\gamma}(t) \quad (31)$$

where $\mathbf{P}$ is a known constant $r \times p$ matrix, $\mathbf{D}$ is a known constant $p \times p$ matrix and the initial condition on $\boldsymbol{\gamma}(t)$ is completely unknown. By proper choice of the eigenvalues and eigen-vectors of $\mathbf{D}$, one can make the response $\mathbf{w}(t)$ of Eqs. (30) and (31) simulate various combinations of a variety of common functions such as steps, ramps, polynominals in t, sinusoids, exponentials, etc. The use of a dynamical model of the form of Eqs. (30) and (31), for the unknown disturbance process $\mathbf{w}(t)$, was proposed in Ref. 39 and more recently used in Ref. 40.

When $\mathbf{w}(t)$ is assumed to be modeled by the dynamical process Eqs. (30) and (31), the solution Eq. (28), obtained previously, can be expressed in a considerably improved format. For this purpose, it is only necessary to adjoin Eq. (31) to Eq. (19) to obtain the composite system

$$\dot{\mathbf{z}} = \left(\frac{\dot{\mathbf{x}}}{\dot{\boldsymbol{\gamma}}}\right) = \left[\begin{array}{c|c} \mathbf{A}(t) & \mathbf{\Sigma P} \\ \hline 0 & \mathbf{D} \end{array}\right]\left(\frac{\mathbf{x}}{\boldsymbol{\gamma}}\right) + \left[\frac{\mathbf{B}}{0}\right](\mathbf{u}) \quad (32)$$

which can be written more compactly, as

$$\dot{\mathbf{z}} = \bar{\mathbf{A}}\mathbf{z} + \bar{\mathbf{B}}\mathbf{u} \quad (33)$$

where

$$\bar{\mathbf{A}} = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{\Sigma P} \\ \hline 0 & \mathbf{D} \end{array}\right], \quad \bar{\mathbf{B}} = \left[\frac{\mathbf{B}}{0}\right] \quad (34)$$

Writing the performance index, Eq. (17) in $\mathbf{z}$ coordinates gives

$$J[\mathbf{u}] = \frac{1}{2}\,\mathbf{z}(T)'\bar{\mathbf{F}}\mathbf{z}(T) + \frac{1}{2}\int_{t_0}^{T} \mathbf{u}(t)'\mathbf{R}(t)\mathbf{u}(t)dt \quad (35)$$

where

$$\bar{\mathbf{F}} = \left[\begin{array}{c|c} \mathbf{F} & 0 \\ \hline 0 & 0 \end{array}\right] \quad (36)$$

It is noted that $\mathbf{F}$ is positive definitive and $\bar{\mathbf{F}}$ is positive semi-definite as required for the existence and uniqueness of the optimal control.[27,29]

According to the general results described previously, the control that minimizes Eq. (35) subject to Eq. (33) and**

$$\mathbf{z}(t_0) = \mathbf{z}_0 = \text{column } (\mathbf{x}_0|\boldsymbol{\gamma}_0)$$
$$\mathbf{z}(T)\text{unspecified};\ t_0, T \text{ specified} \quad (37)$$

is given by,

$$\mathbf{u}^*(t) = -\mathbf{R}(t)^{-1}\bar{\mathbf{B}}'\bar{\mathbf{K}}(t)\mathbf{z}(t) \quad (38)$$

where $\bar{\mathbf{K}}(t)$ satisfies

$$\dot{\bar{\mathbf{K}}} = -\bar{\mathbf{K}}\bar{\mathbf{A}} - \bar{\mathbf{A}}'\bar{\mathbf{K}} + \bar{\mathbf{K}}\bar{\mathbf{B}}\mathbf{R}^{-1}\bar{\mathbf{B}}'\bar{\mathbf{K}};\ \bar{\mathbf{K}}(T) = \bar{\mathbf{F}} \quad (39)$$

Expanding Eq. (38) in original coordinates gives

$$\mathbf{u}^*(t) = -\mathbf{R}(t)^{-1}\bar{\mathbf{B}}'[\bar{\mathbf{K}}_{11}\mathbf{x}(t) + \bar{\mathbf{K}}_{12}\boldsymbol{\gamma}(t)] \quad (40)$$

where

$$\bar{\mathbf{K}} = \left[\begin{array}{c|c} \bar{\mathbf{K}}_{11} & \bar{\mathbf{K}}_{12} \\ \hline \bar{\mathbf{K}}_{12}' & \bar{\mathbf{K}}_{22} \end{array}\right] \quad \begin{array}{ll} \bar{\mathbf{K}}_{11} = n \times n & \bar{\mathbf{K}}_{12} = n \times p \\ & \bar{\mathbf{K}}_{22} = p \times p \end{array}$$

Since the previous result Eq. (28) gives the general solution and Eq. (40) represents the solution for the special case of Eqs. (30) and (31), the two solutions may be equated to obtain the following important relations.

$$\bar{\mathbf{K}}_{11}(t) = \mathbf{K}(t);\ \bar{\mathbf{K}}_{12}(t)\boldsymbol{\gamma}(t) = \mathbf{h}(t) \quad (41)$$

In other words, when $\mathbf{w}(t)$ satisfies the differential Eqs. (30) and (31), the function $\mathbf{h}(t)$ in the optimal control Eq. (28) can be written as a linear function of the *instantaneous value*†† of the variable $\boldsymbol{\gamma}(t)$ as indicated by Eq. (40).

The result that the optimal control depends only on the instantaneous value $\boldsymbol{\gamma}(t)$ [and not on future values of $\boldsymbol{\gamma}(t)$], when Eqs. (30) and (31) hold, could have been anticipated

** Notice that $\boldsymbol{\gamma}_0$, which is related to $\mathbf{w}_0$ by $\mathbf{w}_0 = \mathbf{P}\boldsymbol{\gamma}_0$, is assumed known in the development of the "optimal" control law. On the other hand, it has already been admitted that $\boldsymbol{\gamma}_0$ is, in fact, unknown. In the next section, a scheme for computing an on-line, real-time, estimate $\hat{\boldsymbol{\gamma}}(t)$ of $\boldsymbol{\gamma}(t)$ [assuming $\boldsymbol{\gamma}(t)$ satisfies Eqs. (30) and (31)] will be described. This scheme has the nice property that $\hat{\boldsymbol{\gamma}}(t) \to \boldsymbol{\gamma}(t)$ quickly, regardless of the initial error $\boldsymbol{\gamma}_0 - \hat{\boldsymbol{\gamma}}_0$. Thus, the unknown nature of $\boldsymbol{\gamma}_0$ only serves to introduce a (small) degree of suboptimality in the otherwise optimal control.

†† The instantaneous value of $\boldsymbol{\gamma}(t)$ contains all the information needed to compute $\mathbf{w}(t)$, $d\mathbf{w}(t)/dt$, $d^2\mathbf{w}(t)/dt^2$, etc. In fact, by Eqs. (30) and (31), $d^k\mathbf{w}(t)/dt^k = \mathbf{P}\mathbf{D}^k\boldsymbol{\gamma}(t)$, so that the vector function $\mathbf{w}(t)$ in Eqs. (30) and (31) at least satisfies the sth order, vector differential equation

$$\frac{d^s\mathbf{w}}{dt^s} + \sum_{1}^{s} \frac{\alpha_i d^{i-1}\mathbf{w}}{dt^{i-1}} = 0, \text{ where } \mathbf{D}^s + \sum_{1}^{s} \alpha_i \mathbf{D}^{i-1} = 0$$

is the minimal polynomial of the $p \times p$ matrix $\mathbf{D}$, $s \leqq p$. Of course, the order k of the minimum order equation satisfied by $\mathbf{w}(t)$ will ordinarily be less than s, depending on the null space of $\mathbf{P}$.

from the well-known fact that all future values $\mathbf{w}(\tau)$, $\tau \geqq t$ of Eqs. (30) and (31) are completely determined by the one value $\gamma(t)$, $[\mathbf{w}(\tau) = \mathbf{P}e^{\mathbf{D}(\tau-t)}\gamma(t)]$.

As a practical matter, however, even if $\mathbf{w}(t)$ obeys Eqs. (30) and (31), one might not accurately know to which particular solution of Eqs. (30) and (31) $\mathbf{w}(t)$ corresponds. That is, one might not have accurate knowledge of $\gamma(0)$. In such cases it is desirable to have a scheme for estimating the instantaneous value of $\gamma(t)$ on-line. Such a scheme, which is based on the notion of asymptotic differentiation, is described in the next section.

4.2 An Asymptotic Estimator for the Instantaneous Value $\gamma(t)$ When Eqs. (30) and (31) Hold

In early and unpublished researches, R. W. Bass proposed a notably effective scheme for generating physically realizable, asymptotic estimates of inaccessible states in linear systems. This idea has been exploited more recently by Luenberger,[33] Wolovich,[34] Dellon and Sarachik[35] and others. This approach is directly applicable to the present problem of estimating $\gamma(t)$ since, in the formulation Eq. (32), the elements of the unknown disturbance variable $\gamma(t)$ are considered as auxiliary inaccessible states that augment the original system state vector.

Assume that the system described by Eq. (33) is given and let the output of that system be related to the augmented state $\mathbf{z}$ by the expression

$$\mathbf{y} = \mathbf{C}\mathbf{z}, \quad \mathbf{C} = [\mathbf{I}_{n\times n}|\mathbf{0}_{n\times p}] \tag{42}$$

In other words, $\mathbf{y} = \mathbf{x}$. It can be shown that the initial state $\mathbf{z}(t_0)$ of a completely observable plant of the form Eq. (33) can always be precisely determined, in finite time, from measurement of $\mathbf{u}(t)$ and the output $\mathbf{y}(t)$ over some positive and finite length of time $t_0 < t \leqq t_1$.[29] The particular scheme to be developed here is, essentially, an asymptotic realization of this idea for the determination of $\gamma(t)$ from measurements of $\mathbf{y}(t)$. Therefore it will be necessary to assume hereafter that the augmented state γ is observable‡‡ by measurements of $\mathbf{x}(t)$. This notion is made precise in the Appendix.

To proceed, let $\bar{\mathbf{M}}$ be a real nonsingular $(n + p) \times (n + p)$ matrix and define a nonsingular transformation on $\mathbf{z}$ such that

$$\bar{\xi} = \bar{\mathbf{M}}\mathbf{z} = \bar{\mathbf{M}}(\mathbf{x}/\gamma) \tag{43}$$

Further, partition $\bar{\xi}$ into the two subvectors

$$\bar{\xi} = \bar{\mathbf{M}}(\mathbf{x}/\gamma) = (\xi/\mathbf{y}) \tag{44}$$

and let $\bar{\mathbf{M}}$ be written in the special form

$$\bar{\mathbf{M}} = \left[\frac{\mathbf{M}}{\mathbf{C}}\right] = \left[\begin{array}{c|c}\mathbf{M}_{11} & \mathbf{M}_{12} \\ \hline \mathbf{I}_{n\times n} & \mathbf{0}_{n\times p}\end{array}\right] \tag{45}$$

where $\mathbf{M} = [\mathbf{M}_{11}|\mathbf{M}_{12}]$ is an arbitrary $p\times(n + p)$ matrix, subject to the requirement that $\bar{\mathbf{M}}$ is real and nonsingular. It is easy to establish that

$$\bar{\mathbf{M}}^{-1} = \left[\begin{array}{c|c}\mathbf{0} & \mathbf{I}_{n\times n} \\ \hline \mathbf{M}_{12}^{-1} & -\mathbf{M}_{12}^{-1}\mathbf{M}_{11}\end{array}\right] \tag{46}$$

Using Eqs. (32) and (46) the first derivative of Eq. (43) can be written as

$$\dot{\bar{\xi}} = [\bar{\mathbf{M}}\bar{\mathbf{A}}\bar{\mathbf{M}}^{-1}]\bar{\xi} + \bar{\mathbf{M}}\bar{\mathbf{B}}\mathbf{u} \tag{47}$$

‡‡ The state vector $\mathbf{z}$ in the system Eqs. (33) and (42) is completely observable at $t = t_0$ if, and only if, the matrix $\int_{t_0}^{t_1} \mathbf{\Phi}'(\sigma,t_0)\mathbf{C}'\mathbf{C}\mathbf{\Phi}(\sigma,t_0)d\sigma$ is nonsingular for some finite $t_1 > t_0$, where $\mathbf{\Phi}(t,t_0)$ is the ordinary state transition matrix for $\bar{\mathbf{A}}(t)$.

where $\bar{\mathbf{M}}\bar{\mathbf{A}}\bar{\mathbf{M}}^{-1}$ can be written in the partitioned form

$$\bar{\mathbf{M}}\bar{\mathbf{A}}\bar{\mathbf{M}}^{-1} = \left[\begin{array}{c|c}\mathbf{N}_{11} & \mathbf{N}_{12} \\ \hline \mathbf{N}_{21} & \mathbf{N}_{22}\end{array}\right]$$

$$\begin{aligned}\mathbf{N}_{11} &= \mathbf{M}_{11}\mathbf{\Sigma}\mathbf{P}\mathbf{M}_{12}^{-1} + \mathbf{M}_{12}\mathbf{D}\mathbf{M}_{12}^{-1} \\ \mathbf{N}_{12} &= \mathbf{M}_{11}(\mathbf{A} - \mathbf{\Sigma}\mathbf{P}\mathbf{M}_{12}^{-1}\mathbf{M}_{11}) - \mathbf{M}_{12}\mathbf{D}\mathbf{M}_{12}^{-1}\mathbf{M}_{11} \\ \mathbf{N}_{21} &= \mathbf{\Sigma}\mathbf{P}\mathbf{M}_{12}^{-1} \\ \mathbf{N}_{22} &= \mathbf{A} - \mathbf{\Sigma}\mathbf{P}\mathbf{M}_{12}^{-1}\mathbf{M}_{11}\end{aligned} \tag{48}$$

and where

$$\bar{\mathbf{M}}\bar{\mathbf{B}} = \left[\frac{\mathbf{M}_{11}\mathbf{B}}{\mathbf{B}}\right]$$

Then, Eq. (47) may be written in the subvector form

$$\dot{\xi} = \mathbf{N}_{11}\xi + \mathbf{N}_{12}\mathbf{y} + \mathbf{M}_{11}\mathbf{B}\mathbf{u} \tag{49}$$

$$\dot{\mathbf{y}} = \mathbf{N}_{21}\xi + \mathbf{N}_{22}\mathbf{y} + \mathbf{B}\mathbf{u} \tag{50}$$

The vector $\mathbf{y} = \mathbf{x}$ is the original system state vector which is an assumed known (measured) quantity. The variable ξ represents the unknown quantity to be estimated since, by Eqs. (44) and (46), ξ is a linear combination of the vectors γ and $\mathbf{x}$. Now, let $\hat{\xi}$ denote the vector output of an auxiliary dynamical system defined by [compare with Eq. (49)]

$$\dot{\hat{\xi}} = \mathbf{N}_{11}\hat{\xi} + \mathbf{N}_{12}\mathbf{y} + \mathbf{M}_{11}\mathbf{B}\mathbf{u} \tag{51}$$

where $\mathbf{y}(t)$ and $\mathbf{u}(t)$ are, respectively, the actual measured output and input applied to the system of Eqs. (33) and (42). Suppose $\hat{\xi}(t)$ is a solution to Eq. (51) and assume (for the moment) that $\hat{\xi}(t)$ is an approximation to $\xi(t)$. Then, the corresponding estimate $\hat{\gamma}(t)$ of $\gamma(t)$ is determined by applying the inverse transformation of Eqs. (44) and (46)

$$\hat{\gamma}(t) = \mathbf{M}_{12}^{-1}(\hat{\xi} - \mathbf{M}_{11}\mathbf{x}) \tag{52}$$

It will be shown in the next section that, provided $\mathbf{D}$ and $\mathbf{\Sigma}\mathbf{P}$ satisfy a certain algebraic condition, one can choose $\mathbf{M}_{11}$ and $\mathbf{M}_{12}$ so that the function $\hat{\gamma}(t)$ generated by the auxiliary dynamical system Eqs. (51) and (52) does indeed asymptotically approach the true value $\gamma(t)$ in arbitrarily short settling time. In other words, the auxiliary system Eq. (51), Eq. (52) then constitutes a physically realizable, on-line, asymptotic estimator for computing an accurate estimate of the current value of $\gamma(t)$ [and $\mathbf{w}(t) = \mathbf{P}\gamma(t)$]. A block diagram of the complete suboptimal momentum desaturation feedback control system using the controller Eq. (40) and the disturbance estimator Eqs. (51) and (52), is shown in Fig. 6.

4.3 Properties of the Asymptotic Estimator for $\gamma(t)$

If the block $\mathbf{M}_{12}$ in the $p \times (n + p)$ matrix $\mathbf{M}$ in Eq. (45) is selected to have the special form $\mathbf{M}_{12} = \mathbf{I}_{p\times p}$ then Eq. (45) takes the special form

$$\bar{\mathbf{M}} = \left[\begin{array}{c|c}\mathbf{M}_{11} & \mathbf{I}_{p\times p} \\ \hline \mathbf{I}_{n\times n} & \mathbf{0}\end{array}\right] \tag{53}$$

and it is readily verified that $\bar{\mathbf{M}}$ is always nonsingular regardless of the choice of $\mathbf{M}_{11}$. In fact, by Eq. (46), $\bar{\mathbf{M}}^{-1}$ is then given explicitly by

$$\bar{\mathbf{M}}^{-1} = \left[\begin{array}{c|c}\mathbf{0} & \mathbf{I}_{n\times n} \\ \hline \mathbf{I}_{p\times p} & -\mathbf{M}_{11}\end{array}\right] \tag{54}$$

and the blocks in the partitioned matrix $\mathbf{N}$, in Eq. (48), are

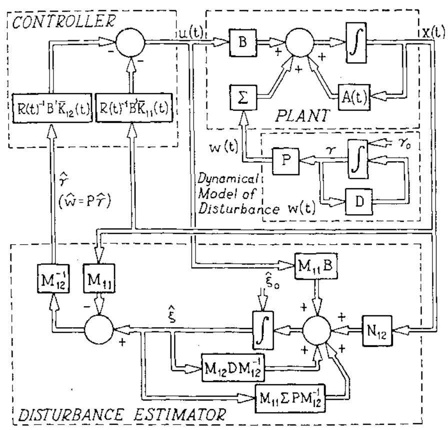

Fig. 6 Suboptimal closed loop system using on-line asymptotic estimator for the unknown disturbance variable $\gamma(t)$.

given by

$$\mathbf{N}_{11} = \mathbf{D} + \mathbf{M}_{11}\mathbf{\Sigma P};\ \mathbf{N}_{12} = \mathbf{M}_{11}(\mathbf{A} - \mathbf{\Sigma P M}_{11}) - \mathbf{DM}_{11}$$
$$\mathbf{N}_{21} = \mathbf{\Sigma P};\ \mathbf{N}_{22} = \mathbf{A} - \mathbf{\Sigma P M}_{11} \quad (55)$$

The effectiveness of the estimator Eqs. (51) and (52) in this case is clearly demonstrated by observing how the error

$$\mathbf{e} = \boldsymbol{\gamma} - \hat{\boldsymbol{\gamma}} \quad (56)$$

propagates with time. It is straightforward to show that the first derivative of Eq. (56) may be written in the homogeneous form

$$\dot{\mathbf{e}} = (\mathbf{D} + \mathbf{M}_{11}\mathbf{\Sigma P})\mathbf{e} \quad (57)$$

Thus, under the particular choice $\mathbf{M}_{12} = \mathbf{I}$, if the completely arbitrary $p \times n$ matrix $\mathbf{M}_{11}$ can be chosen so that Eq. (57) is asymptotically stable with arbitrarily short settling time, then the solution $\hat{\boldsymbol{\gamma}}(t)$ of Eqs. (51) and (52) always quickly approaches $\boldsymbol{\gamma}(t)$ [likewise $\hat{\mathbf{w}}(t)$ quickly approaches $\mathbf{w}(t)$ since $\mathbf{w} = \mathbf{P}\boldsymbol{\gamma}$]. This result obtains for any nonsingular $\mathbf{M}_{12}$ since $\mathbf{M}_{12}$ only acts as a scale factor.

It turns out that one can indeed choose $\mathbf{M}_{11}$ so that Eq. (57) is asymptotically stable with arbitrarily short settling time if, and only if, the matrices $\mathbf{D}$ and $\mathbf{\Sigma P}$ satisfy the algebraic condition

$$\text{rank}\ [\mathbf{P}'\mathbf{\Sigma}'|\mathbf{D}'\mathbf{P}'\mathbf{\Sigma}'|\mathbf{D}'^2\mathbf{P}'\mathbf{\Sigma}'|\ \ldots\ |\mathbf{D}'^{(p-q)}\mathbf{P}'\mathbf{\Sigma}'] = p \quad (58)$$

where $q = \text{rank}\ (\mathbf{\Sigma P})$. The detailed derivation of Eq. (58), and several effective algorithms for constructing an appropriate matrix $\mathbf{M}_{11}$, are presented in the Appendix. Thus, assuming that Eq. (58) is satisfied and that $\mathbf{M}_{11}$ has been appropriately chosen, one can deduce from Eq. (57) the following important physical properties of the estimator.

1) The estimate $\hat{\boldsymbol{\gamma}}(t)$ will quickly converge to $\boldsymbol{\gamma}(t)$ regardless of the initial error $(\boldsymbol{\gamma}_0 - \hat{\boldsymbol{\gamma}}_0)$. This means that the selection of the initial conditions $\hat{\boldsymbol{\xi}}_0$ of the estimator equation Eq. (51) is not critical (i.e., may always be set to zero, if convenient). For the special case where $\hat{\boldsymbol{\xi}}_0 = \boldsymbol{\xi}_0$ the system performance is theoretically optimal at every instant of time.

2) The convergence of the estimate $\hat{\boldsymbol{\gamma}}(t)$ is independent of the control policy $\mathbf{u}(t)$. Since $\mathbf{u}(t)$ is assumed to be also an input to the estimator, it does not matter how $\mathbf{u}(t)$ is generated.

3) The stability properties of the estimator are independent of the stability properties of the plant. Moreover, the rapid asymptotic convergence of $\hat{\boldsymbol{\gamma}}(t)$ to $\boldsymbol{\gamma}(t)$ is assured even in the presence of periodic and/or unstable disturbances ($\mathbf{D}$ may have eigenvalues with zero or positive real parts).

4) The order of the auxiliary dynamical estimator system Eqs. (51) and (52) is exactly the same as the order p of the dynamical disturbance process Eqs. (30) and (31) which is assumed to model $\mathbf{w}(t)$.

If the condition Eq. (58) happens not to be satisfied, then it is not possible to choose $\mathbf{M}_{11}$ (or $\mathbf{M}_{12}$) so as to make $\mathbf{e}(t) \to \mathbf{0}$ with arbitrarily short settling time. In this case, one cannot depend on $\hat{\boldsymbol{\gamma}}(t)$ always approaching $\boldsymbol{\gamma}(t)$ sufficiently fast. However, in that event, it might still be possible to choose $\mathbf{M}_{11}$ so that Eq. (57) is at least asymptotically stable. That is, at least $\mathbf{e}(t) \to \mathbf{0}$ as $t \to \infty$. The corresponding auxiliary system Eqs. (51) and (52) would then produce an output $\hat{\boldsymbol{\gamma}}(t)$ which at least "approaches" $\boldsymbol{\gamma}(t)$ for sufficiently large times t. Although sluggish in nature, such an estimator might still be useful in certain specific applications where the operating time $(T - t_0)$ is relatively long. A detailed analysis of the precise necessary and sufficient conditions for the existence of a matrix $\mathbf{M}_{11}$ which makes Eq. (57) asymptotically stable, is presented in the Appendix.

An optimization and disturbance cancellation problem related to, but slightly different from, the one treated here has recently been described in Refs. 36 and 40. The technique proposed in Refs. 36 and 40, utilizes higher order derivatives of $\mathbf{u}$ in the performance index (instead of $\mathbf{u}$ as in the present problem) and requires that the control $\mathbf{u}(t)$ have the capability to exactly cancel the disturbance $\mathbf{w}(t)$. (There must exist a $\mathbf{u}$ such that $\mathbf{Bu} = -\mathbf{\Sigma w}$ for all $\mathbf{w}$; see footnote 4 of Ref. 36.) This latter condition is necessary in order to achieve $\mathbf{x}(t) \to \dot{\mathbf{x}}(t) \to \mathbf{0}$ as $t \to \infty$ as required in the problem formulation in Refs. 36 and 40. In the particular problem treated in the present work, however, no such control exists [examine Eqs. (12) and (13)] and indeed if it did, there would be no momentum accumulation (desaturation) problem. Moreover, in that latter event, the general disturbance estimation-feedback control scheme developed here (and depicted in Fig. 6) would be similar to that described in Ref. 40.

5. An Application: The NASA Apollo Telescope Mount (ATM)

In the NASA space project called Apollo Telescope Mount (ATM), the sun is to be studied by means of an Earth-orbiting manned space station. Some eight different scientific experiments are to be performed by instruments mounted on a gimbaled spar attached to the space station. This spar is pointed to desired points on the solar disk by an Experiment Pointing Control system (EPC). The spar itself is gimbaled from the main carrier vehicle which is an S-IVB stage equipped as a workshop. The carrier vehicle attitudes are controlled by a CMG system of the type discussed in the Introduction of this paper. The total configuration is sketched in Fig. 4.

During experimentation periods (the so-called daylight mode of operation) the vehicle axis v_3 should point to the sun while the v_1 and v_2 axes may be pointed in any fixed direction. These latter directions are to be dictated by other considerations such as the minimization of the degree of saturation in the CMG system. The particular daylight flying attitudes θ_d which minimize the degree of saturation buildup in the CMG system while allowing the v_3 axis of Fig. 4 to a point along the sun line, have been derived in Ref. 38.

The choice of the parameter $\boldsymbol{\delta}$ in Eq. (6) determines the mean of the ATM controller momentum over many orbits. Techniques for selecting $\boldsymbol{\delta}$ with only gravity gradients acting are discussed in Ref. 25.

In the so-called night mode of operation of the ATM, the sun is occulted by the Earth and no sun pointing constraints are placed on the vehicle. This period of flight can be used to carry out the momentum desaturation of the CMG controllers as discussed above in Section 3 (in the case of known disturbances) and Section 4 (in the case of unknown disturbances) of the present paper. For instance, Fig. 5 illustrates an optimal control configuration for ATM momentum desaturation when the disturbances $\mathbf{\Sigma w}(t)$ are assumed known (as is appropriate when gravity-gradient torques are the predominant disturbances).

Aerodynamic torques would be included in the external torques $\mathbf{T}_{ex}$ in Eq. (1) when they are significant. In this case, one would write the aerodynamic torques as a truncated Fourier series as in Ref. 37. Small angle approximations in the variables θ_i would then be made in the analytical expression for $\mathbf{T}_{ex}(t)$ which includes gravity gradient and aerodynamic torques. This would give an equation of motion as in Eq. (5) except $\mathbf{T}_n(t)$ and $\mathbf{G}(t)$ will contain, in addition to the terms already defined from gravity gradients, the Fourier series terms from the aerodynamics. Then, a control law of the form of Eq. (28) can be constructed according to Section 3, where the definitions for $\mathbf{T}_n(t)$ and $\mathbf{G}(t)$ are as noted above. Such a law would be optimal if the explicit mathematical function used to describe the disturbances was exact.

However, the inevitable uncertainties and fluctuations in atmospheric density and other variables at orbital altitudes prompts one to consider the alternative disturbance estimating control law (Fig. 6) that will adapt to such uncertainties. For this purpose, one must choose the disturbance modeling matrices $\mathbf{D}$ and $\mathbf{P}$, such that Eqs. (30) and (31) are satisfied for the $\mathbf{T}_n(t)$ previously defined [see text following Eq. (10) for the relation between $\mathbf{T}_n$ and $\mathbf{w}$]. Then, the estimator Eqs. (51) and (52) will adapt to any magnitude and waveshape of the actual $\mathbf{T}_n(t)$ if no new modal components are present in the actual $\mathbf{T}_n(t)$ that were not modeled in the analytical expression of $\mathbf{T}_n(t)$.

References 20–26 document the previously proposed schemes for desaturating, and otherwise managing, vehicle and controller momentum in the example ATM system. The method proposed in this section of the present paper differs from that of prior investigators in the following ways: 1) the approach is systematic, using state variables and modern control theory; 2) no special vehicle maneuver waveforms are specified a priori, and no special assumptions are made about vehicle inertia distributions§§; 3) the initial time t_0 and the terminal time T of the desaturation process can be arbitrarily chosen; 4) a physically meaningful mathematical performance criterion is identified for which the system is optimized; 5) the controller is given in feedback form (Refs. 20 and 21 also have this property); 6) an effective technique is described for accommodating a particular class of unknown disturbances by means of a disturbance estimator. Some simulation results comparing the method presented here with other methods of desaturating the CMG controller in the ATM system are currently under study.

6. Conclusions

In this study, a rather general CMG momentum desaturation control law has been derived that makes optimal use of the natural external environmental torques acting on the vehicle. One case is treated where the external disturbing torques are assumed completely known a priori. For that case, the optimal control law is a linear function of measured states and requires that position and momentum errors be fed back through specified time-varying gains. A second case, where the external disturbance torques may vary, or otherwise be uncertain, has also been considered. For this latter case, the proposed controller automatically adapts to the unknown disturbance torques and keeps over-all performance near-optimal. This controller requires, in addition to the specified time-varying gains, an additional dynamic feedback loop that can be viewed as a real-time, on-line estimator for determining the unknown disturbance function $\mathbf{w}(\tau), t \leqq \tau \leqq T$, from measurements of the plant state $\mathbf{x}(t)$.

As an illustration of the general theory developed here, it has been shown how one might apply the proposed method to the NASA ATM project. The resulting desaturation control law appears to offer several advantages over previously proposed schemes.

A topic not discussed in this paper is the rational selection of $\boldsymbol{\sigma}$, the desired terminal momentum $\mathbf{H}_c(T)$ of the desaturated controller. This parameter, in effect, serves to define ideal desaturation, since $\boldsymbol{\sigma}$ is directly related to the gimbal angles of the CMG controller. The best choice of $\boldsymbol{\sigma}$ will, of course, depend on the characteristics of a particular application and will involve consideration of such things as the nature of the mission, disturbance torques, etc.

Appendix

1. Necessary and Sufficient Condition that Eq. (57) Can Be Stabilized with Arbitrarily Short Settling Times

In order for the auxiliary dynamical system Eqs. (51) and (52) to qualify as a legitimate estimator of¶¶ $\gamma(t)$ it is essential that $\hat{\gamma}(t)$ always quickly approach $\gamma(t)$, regardless of the initial conditions $\hat{\gamma}(t_0), \gamma(t_0)$. Just how quickly this must occur will depend, of course, on the given terminal time and the nature of the function $\gamma(t)$ in each particular problem. However, to be generally capable of accommodating any set of given problem parameters, it is necessary that one be able to choose M_{11} so that $\hat{\gamma}(t) \to \gamma(t)$ within any prespecified (arbitrarily small) settling time. Since the spectrum of D is completely unrestricted (except that complex eigenvalues of D must occur in conjugate pairs) it follows from Eq. (57) that the desired influence over settling time can be achieved if, and only if, one can choose M_{11} so that the real part of every eigenvalue of $D + M_{11}\Sigma P$ is less than some arbitrarily chosen large negative number. It is not difficult to show that this latter property is entirely equivalent to the property that one can choose M_{11} so as to realize *any* arbitrary prespecified spectrum for $D + M_{11}\Sigma P$. It is recalled that a matrix and its transpose share the same spectrum. Thus, for mathematical convenience, we will hereafter work with the transposed expression $(D + M_{11}\Sigma P)' = D' + P'\Sigma'M'_{11}$.

Wonham has recently shown, in Ref. 41, that every prespecified spectrum for $L + NM$, $L = n \times n$, $N = n \times r$, $M = r \times n$, can be realized by proper choice of M if, and only if, the given pair (L,N) is completely controllable in the sense of Kalman. That is, if, and only if,

$$\operatorname{rank}[N|LN|L^2N| \ldots |L^{(n-q)}N] = n \qquad \text{(A1)}$$

where $q = \operatorname{rank} N$. Translating this result in terms of $D' + P'\Sigma'M'_{11}$, and using the aforementioned properties, we have the first main result of this Appendix.

Theorem A-1

The matrix M_{11} in Eq. (57) can be chosen so that the real part of every eigenvalue of $D + M_{11}\Sigma P$ is less than some arbitrarily chosen large negative number ($\to -\infty$) if, and only if,

$$\operatorname{rank}\,[P'\Sigma'|D'P'\Sigma'|D'^2P'\Sigma'| \ldots |D'^{(p-q)}P'\Sigma'] = p \qquad \text{(A2)}$$

where $q = \operatorname{rank}\,(\Sigma P)$.

§§ The desaturating maneuver angles should be small for the linear optimization techniques presented here to be valid. Since desaturation can only be accomplished if the polarity of the environmental torques are reversed by the maneuvers, there may exist situations involving large displacements between principal inertia axes (about which gravity-gradient torques act) and the geometric axes (from which desired pointing attitudes are determined) for which torque reversal is not possible with small angles.

¶¶ Through an oversight of the authors, most of the vectors and matrices in this Appendix were not set in Boldface.

In Kalman's terminology, the condition (A2) is said to correspond to the condition that the pair $(D,\Sigma P)$ is "completely observable." Physically, this implies that every eigenmode of $\dot{\mathbf{\gamma}} = \mathbf{D\gamma}$ can be observed by monitoring the vector $\mathbf{\Sigma w}(t), \mathbf{w} = \mathbf{P\gamma}$.

2. Effective Computation of a Stabilizing Matrix M_{11} When Eq. (58) Holds

The construction of a matrix M_{11} so that Eq. (57) has arbitrarily small settling times [when Eq. (58) or Eq. (A2) holds] can be accomplished in several ways. One method consists of formulating the search for M_{11} as a classical linear-quadratic optimization problem.[29] In this way, M_{11} is determined by the expression

$$M_{11} = -\tfrac{1}{2}\pi P'\Sigma' \tag{A3}$$

where $\pi = \pi' > 0$ is the unique, symmetric, positive definite solution of the matrix quadratic equation

$$\pi(D' + \eta I) + (D + \eta I)\pi - \pi P'\Sigma'\Sigma P\pi + Q = 0 \tag{A4}$$

Q is an arbitrarily selected, real, symmetric, positive definite matrix, and η is a nonpositive scalar constant. The resulting eigenvalues of $D + M_{11}\Sigma P$ [and $D' + P'\Sigma'M'_{11}$], with M_{11} given by Eq. (A3), will all have real parts less* than η, but their precise location is not known a priori since that eigenvalue pattern is determined, in a complex and indirect way, by the choice of the matrix Q. It is remarked that a considerable simplification of this stabilization procedure has recently been proposed by B. Porter.[42,43] However, it turns out that Porter's method is not reliable,[44] owing to certain misconceptions in his derivation.

As an alternative procedure, one can synthesize the stabilizing matrix M_{11} by using the more precise pole assignment method of Wonham.[41] In this way, the desired eigenvalues of $(D' + P'\Sigma'M'_{11})$ are arbitrarily preselected beforehand (based on the control designers choice for the desired settling times, etc.) and the required matrix M_{11}' is then computed. Several effective schemes for carrying out this latter step are described in Ref. 45.

3. Necessary and Sufficient Condition that Eq. (57) Can at Least Be Stabilized

As pointed out in Section 4.3, if it turns out that Eq. (58) [or Eq. (A2)] is not satisfied, it may still be desirable to attempt to choose M_{11} so that Eq. (57) is at least asymptotically stable. For this purpose, it is useful to know the necessary and sufficient conditions under which such a stabilizing matrix M_{11} actually exists.

Stabilization of the system Eq. (57), consists of choosing a real, $p \times n$ matrix M_{11} such that every eigenvalue of $(D + M_{11}\Sigma P)$ has a strictly negative real part. In this process, the matrices (Σ,P,D) defined in Eqs. (12, 30, and 31) respectively, are all assumed fixed. Intuitively, it seems plausible that a stabilizing matrix M_{11} should exist if at least the unstable eigenmodes of D are completely observable in the sense of Kalman. Here, it will be shown that this notion, when made precise, is indeed correct.

It is recalled that the stabilization of $(D + M_{11}\Sigma P)$ is entirely equivalent to the stabilization of the transposed expression $(D' + P'\Sigma'M'_{11})$. In order to take advantage of certain mathematical simplifications, we hereafter consider the equivalent problem of choosing M'_{11} to stabilize $(D' + P'\Sigma'M'_{11})$. Moreover, since Σ,P are fixed, we will write $\Sigma P = G$. To begin, it is first necessary to define the stable and unstable eigenspaces (modal subspaces) of D'.

* Recall that, in the complex plane, the spectrum of $A + \eta I$, $\eta \leqq 0$, is the spectrum of A shifted horizontally to the left by the amount $-\eta$.

Definition

Let $\alpha(\lambda)$ be the minimum polynominal of the real $p \times p$ matrix D' and suppose $\alpha(\lambda)$ is factored into two coprime polynomials

$$\alpha(\lambda) = \alpha^+(\lambda)\alpha^-(\lambda) \tag{A5}$$

where all zeros of $\alpha^+(\lambda)$ lie in the closed right-half complex plane and all zeros of $\alpha^-(\lambda)$ lie in the open left-half complex plane. Further, let the two (linear) subspaces E^+ and E^- be defined as

$$E^+ = \{x|\alpha^+(D')x = 0, x\epsilon E^p\} = p^+\text{-dimensional subspace} \tag{A6}$$

$$E^- = \{x|\alpha^-(D')x = 0, x\epsilon E^p\} = p^-\text{-dimensional subspace} \tag{A7}$$

$$p^+ + p^- = p$$

Then $E^+(E^-)$ is the subspace of unstable (stable) modes of the matrix D'. Note that $E^+(E^-)$ are D'-invariant subspaces.

Next, it is necessary to identify the projection of an arbitrary $x\epsilon E^p$ onto the subspaces E^+ and E^-. For this purpose, we proceed as in Ref. 46 and let the $p \times p$ matrix $P^+(P^-)$ be the projection operator projecting x onto $E^+(E^-)$ along $E^-(E^+)$ (for the theory of projection operators, see Ref. 47) and define

$$x^+ = P^+x = \text{projection of } x \text{ on } E^+ \text{ along } E^- = p \text{ vector} \tag{A8}$$

$$x^- = P^-x = \text{projection of } x \text{ on } E^- \text{ along } E^+ = p \text{ vector} \tag{A9}$$

Finally, let the p vectors x^+ and x^- be represented on the subspaces E^+ and E^- in terms of suitable p^+-dimensional and p^--dimensional bases erected on E^+ and E^-. Then, letting (y,z) denote the image of (x^+,x^-) referred to those bases, one can write

$$x^+ = R^+y;\ x^+\epsilon E^+ \tag{A10}$$

$$x^- = R^-z;\ x^-\epsilon E^- \tag{A11}$$

where $R^+(R^-)$ is any real maximal rank $p \times p^+(p \times p^-)$ matrix whose columns form a basis for $E^+(E^-)$.

Now it will be shown that, by means of an appropriate similarity transformation, the underlying algebraic relationship between M_{11} and the unstable and stable modes of D' can be clearly exhibited. For this purpose, consider the nonsingular linear transformation

$$x = S(y/z);\ y = p^+ - \text{vector};\ z = p^- - \text{vector} \tag{A12}$$

acting on the pth-order dynamical system [compare with Eq. (57)]

$$\dot{x} = D'x + G'\mu \tag{A13}$$

$$\mu = M'_{11}x \tag{A14}$$

where the p vector x in Eqs. (A6 and 14) is a dummy variable [not to be confused with the state vector x in Eq (12)] and S is a $p \times p$ matrix defined by

$$S = [R^+|R^-] \tag{A15}$$

It is easy to establish the

Lemma

$$S^{-1} = \left[\frac{(R^{+\prime}R^+)^{-1}R^{+\prime}P^+}{(R^{-\prime}R^-)^{-1}R^{-\prime}P^-}\right] \tag{A16}$$

Proof of the lemma

Since S is clearly nonsingular, it follows that S^{-1} exists and is unique. Thus, it suffices to compute

$$S^{-1}S = \left[\frac{(R^{+\prime}R^{+})^{-1}R^{+\prime}P^{+}}{(R^{-\prime}R^{-})^{-1}R^{-\prime}P^{-}}\right][R^{+}|R^{-}] \quad \text{(A17)}$$

to obtain

$$S^{-1}S = \left[\begin{array}{c|c} I_{p^{+}\times p^{+}} & 0 \\ \hline 0 & I_{p^{-}\times p^{-}} \end{array}\right] \quad \text{(A18)}$$

where we have used the fact that

$$P^{+}R^{+} = R^{+},\ P^{+}R^{-} = 0$$
$$P^{-}R^{+} = 0,\ P^{-}R^{-} = R^{-} \quad \text{(A19)}$$

Corollary

$$R^{+}(R^{+\prime}R^{+})^{-1}R^{+\prime}P^{+} + R^{-}(R^{-\prime}R^{-})^{-1}R^{-\prime}P^{-} = I \quad \text{(A20)}$$

Using Eqs. (A15) and (A16), it is found that the transformation Eq. (A12) takes Eqs. (A13) and (A14) into the form

$$\left(\frac{\dot{y}}{\dot{z}}\right) = \left[\begin{array}{c|c} A^{+} & O \\ \hline O & A^{-} \end{array}\right]\left(\frac{y}{z}\right) + \left[\frac{B^{+}}{B^{-}}\right](\tilde{\mu}) \quad \text{(A21)}$$

$$\tilde{\mu} = M'_{11}S\left(\frac{y}{z}\right) \quad \text{(A22)}$$

where

$$A^{+} = (R^{+\prime}R^{+})^{-1}R^{+\prime}D'R^{+} \quad \text{(A23)}$$

$$A^{-} = (R^{-\prime}R^{-})^{-1}R^{-\prime}D'R^{-} \quad \text{(A24)}$$

$$B^{+} = (R^{+\prime}R^{+})^{-1}R^{+\prime}P^{+}G',\ G = \Sigma P \quad \text{(A25)}$$

$$B^{-} = (R^{-\prime}R^{-})^{-1}R^{-\prime}P^{-}G' \quad \text{(A26)}$$

Expressions (A23–A26) follow from the fact that E^{+} and E^{-} are D' invariant, together with the fact that $P^{+}x^{+} = x^{+}, P^{-}x^{-} = x^{-}$, and $P^{+}x^{-} = P^{-}x^{+} = 0$, for all $x^{+}\epsilon E^{+}$, $x^{-}\epsilon E^{-}$.

We are now in a position to state the second main result of this Appendix as Theorem A-2.

Theorem A-2

A real $p \times n$ matrix M_{11}, which stabilizes $(D + M_{11}\Sigma P)$, exists if, and only if, the pair (A^{+},B^{+}) in Eqs. (A23) and (A25) is completely controllable in the sense of Kalman [see (A1)].

Proof of Theorem A-2

To prove theorem A-2's sufficiency, it is only necessary to cite the well-known Fundamental Stabilization Theorem for Linear Dynamical Systems[48] which, for the particular system Eq. (A21), states that: if the pair (A^{+},B^{+}) is completely controllable, then there always exists a matrix C^{+} such that $(A^{+} + B^{+}C^{+})$ is stabilized. It follows that Eqs. (A21) and (A22) [and therefore Eq. (57) as well] is stabilized by the choice

$$M'_{11} = C^{+}(R^{+\prime}R^{+})^{-1}R^{+\prime}P^{+} \quad \text{(A27)}$$

To prove necessity, suppose that the pair (A^{+},B^{+}) is not completely controllable. Then, since every eigenvalue of A^{+} has a nonnegative real part it follows that, for every time function $\tilde{\mu}(t) = \phi(t)$, the corresponding solution of Eq. (A21) satisfies

$$\lim_{t\to\infty} \|y[t;y_0,\phi(t)]\| \neq 0,\ \forall y(t_0) = y_0 \neq 0 \quad \text{(A28)}$$

This completes the proof of theorem A-2.

Theorem A-2 is satisfied, in particular, if $p^{+} = 0$ (all eigenvalues of D have negative real parts), in which case one can achieve stabilization of Eq. (57) by the trivial choice $M_{11} = 0$. The condition that (A^{+},B^{+}) be completely controllable is entirely equivalent to the condition that all eigenmodes of the unstable subsystem $\dot{\gamma} = \tilde{D}\gamma, \tilde{D} = A^{+\prime}$, are observable by monitoring the vector quantity $\Sigma\tilde{w}(t), \tilde{w}(t) = P^{+\prime}R^{+}(R^{+\prime}R^{+})^{-1}\gamma$.

The construction of a matrix M_{11} which will stabilize Eq. (57) (when Eq. (58) is not satisfied) can proceed in exactly the same manner as before. Thus, to use the quadratic optimization method, one simply replaces D' and $P'\Sigma'$ in Eq. (A4) by A^{+} and B^{+}, respectively. The corresponding matrix C^{+} is then given by $C^{+} = -\frac{1}{2}B^{+\prime}\pi$ and, finally, the sought matrix M_{11} is given by the transpose of Eq. (A27). As an alternative method, one can use Wonham's pole-assignment method on the system $A^{+} + B^{+}C^{+}$ to determine C^{+}. Then M_{11} is given, as before, by Eq. (A27) transposed. In any event, the eigenvalues of the original matrix $N_{11} = (D + M_{11}\Sigma P)$ will, of course, consist of the stabilized eigenvalues of $A^{+} + B^{+}C^{+}$ together with the (originally) stable eigenvalues of the matrix A^{-}.

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